

# A note on impossibility of uniformly non-oscillatory approximation \*

Ritesh Kumar Dubey<sup>†</sup>

Research Institute, SRM University, Tamilnadu  
India

May 27, 2015

## Abstract

In this note we show that it is impossible to have data independent non-oscillatory three point finite difference scheme irrespective of its accuracy for a scalar hyperbolic initial value problem.

**keywords** Induced oscillations and numerical stability; affine combination; smoothness parameter; finite difference schemes; transport equation.

**AMS Classification:** 65M12, 35L65, 35L67, 35L04, 65M06, 5M22.

## 1 Introduction

It is well elaborated in literature that evolution of discontinuities are inevitable in the solution of scalar hyperbolic conservation laws and artificial induced oscillations may appear in a numerical approximation in the vicinity of discontinuity [2, 3, 13]. One of the main emphasis in the numerical approximation of hyperbolic conservation laws is to construct schemes which are capable of yielding non-oscillatory approximation [8, 9, 11] see also [10]. The numerical oscillations by high order schemes are well understood e.g., using modified equation analysis. In [1], Lax showed that induced oscillations analogous to the Gibbs phenomena must be present when the solution is approximated by a difference scheme that is more than first order accurate. Recently the cause of induced oscillations in the solution even by first order monotone finite difference scheme for hyperbolic problem is investigated in [4, 5, 6, 7]. This work starts with one motivational example to show that induced oscillations depend on the initial data. We define the notion of data dependent stability and uniformly non-oscillatory approximation. We finally show that it is impossible to have uniformly non-oscillatory approximation by any three point scheme irrespective of its accuracy.

## 2 Uniformly non-oscillatory approximation

We consider the following simple linear initial value problem,

$$\frac{\partial u(x, t)}{\partial t} + a \frac{\partial u(x, t)}{\partial x} = 0, a \neq 0, u(x, 0) = u_0(x) \quad (1)$$

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\*Carried out work is financially supported by DST India through project # SR/FTP/MS-015/2011

<sup>†</sup>mail-to: riteshkd@gmail.com; riteshkumar.d@res.srmuniv.ac.in

where  $u(x, t)$  is a scalar function of space  $x$  and time  $t$  variables, and  $u_0(x)$  is a piece-wise smooth function with possible discontinuities, and characteristic function  $a(x, t)$  is a smooth function. We discretize the space and time variable in to a computational mesh using spatial and temporal step  $h$  and  $k$  respectively with grid points  $(x_i = ih, t_n = nk)$ . Let  $\lambda = \frac{k}{h}$  and  $u_i^n$  represents and approximation of the value  $u(ih, nk)$ .

## 2.1 Lax-Wendroff (LxW) scheme

Consider LxW scheme for (1). It is second order accurate therefore introduces oscillations in computed solution for discontinuous initial condition and is well understood by [1]. We now consider, (1) corresponding to smooth initial condition

$$\begin{aligned} \text{a } u_0(x) &= \sin(\pi x), \quad x \in [-1 : 1] \\ \text{b } u_0(x) &= \begin{cases} \exp\frac{-1}{1-x^2} & \text{if } x \in [-1 : 1] \\ 0 & \text{else} \end{cases} \end{aligned}$$

The numerical results by LxW are given in Figure 1. It is clear from the results that even though both initial conditions are smooth, LxW does not introduces oscillations in solution corresponding to initial condition (a) while oscillations appears in the solution for initial data (b). Also as mentioned above even a first order monotone Lax-Friedrichs scheme exhibits local oscillations for discontinuous solution see [5]. Therefore it can be concluded that the induced oscillations depends on the initial data and can induce in to a numerical solution irrespective of its accuracy. This makes it reasonable to analyze the oscillatory behavior of any scheme with respect to the initial data.

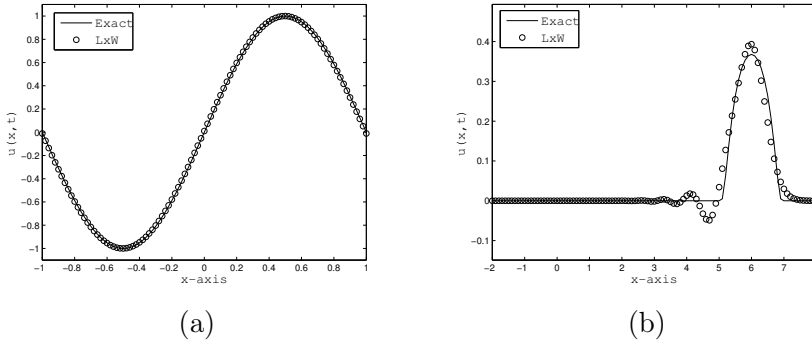


Figure 1: Solution using data  $CFL = 0.8, t = 6, N = 100$ . Both initial condition are with out discontinuity (a) No induce oscillation (b) induced oscillations

## 2.2 A non-oscillatory condition for two point

The idea is to use the following simple concept from geometry: Let each  $P_1, P_2$  denotes two points in a finite affine plane then there affine combination is defined by

$$P = P_1 + \delta(P_2 - P_1) \quad (2)$$

Geometrically,  $P$  represents a point on the line  $L$  which passes through  $P_1$  and  $P_2$ , and if  $0 \leq \delta \leq 1$ ,  $P$  lies on the line segment joining point  $P_1$  and  $P_2$ . Then following the method of characteristics and

affine combination (2), an initial data independent non-oscillatory solution of (1) can be ensured by an explicit two point consistent difference schemes scheme of the form

$$u_j^{n+1} = \begin{cases} u_j^n - D\Delta u_j^n, & \text{if } a(x, t) > 0, \\ u_j^n + D\Delta_+ u_j^n, & \text{if } a(x, t) < 0, \end{cases} \quad (3)$$

provided  $0 \leq D \leq 1$ . The coefficient  $D$  depends on  $a$ ,  $\lambda$ . A simple Von-Neumann stability analysis also shows that (3) is stable provided  $D \leq 1$ . We call scheme (3) uniformly non-oscillatory (UNO).

### 3 Non-oscillatory condition for three point scheme

In this section we show that it is impossible to have a initial data independent uniformly non-oscillatory scheme with higher then two points. In [1], Lax pointed out the use of the following smoothness parameter to devise hybrid scheme which can avoid the Gibbs phenomena. It is defined similar to the one in [17, 12, 14]

$$\theta_i^n = \begin{cases} \theta_i^{n,+} = \frac{\Delta_- u_i^n}{\Delta_+ u_i^n} & \text{if } a \geq 0 \\ \theta_i^{n,-} = \frac{\Delta_+ u_i^n}{\Delta_- u_i^n} & \text{if } a < 0. \end{cases} \quad (4)$$

Using above affine combination we define

**Definition 3.1.** Consider a consistent three point scheme written in the form

$$u_j^{n+1} = \begin{cases} u_j^n + D(a\lambda; \theta_i)\Delta_- u_j^n & \text{if } a(x, t) > 0 \\ u_j^n + D(a\lambda; \theta_i)\Delta_+ u_j^n & \text{if } a(x, t) < 0, \end{cases} \quad (5)$$

where  $i \in \{j, j \pm 1\}$  and the parameter coefficient  $D$  depends on CFL number  $a\lambda$  as well smoothness parameter (4). The scheme (5) is uniformly non-oscillatory (UNO) if

$$0 \leq D(\theta_i) \leq 1, \quad \forall \quad \theta_i \in \mathbb{R}, \quad (6)$$

In case (6) holds only for  $\theta_i \in \mathbb{R} \setminus \mathbb{S}$  where  $\mathbb{S} \subset \mathbb{R}$ , scheme (5) is called Data dependent stable (DDS). Note that three point scheme (5) using a condition on  $\theta$  ensures that the updated value  $u_i^{n+1}$  satisfies the following local maximum principle

$$\begin{aligned} u_{i-1}^n &\leq u_i^{n+1} \leq u_i^n, & \text{if } a > 0, \\ u_i^n &\leq u_i^{n+1} \leq u_{i+1}^n, & \text{if } a < 0. \end{aligned}$$

#### 3.1 Three point upwind linear schemes

A generic numerical flux function for any linear three point upwind scheme for (1) can be given by,

$$F_{i+\frac{1}{2}} = \begin{cases} \alpha u_i^n + \beta u_{i-1}^n & \text{if } a > 0 \\ \alpha u_{i+1}^n + \beta u_{i+2}^n & \text{if } a < 0. \end{cases} \quad (7)$$

where for consistency the coefficients must satisfy  $\alpha + \beta = a$ .<sup>1</sup>

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<sup>1</sup>A consistent discretization requires i.e.,  $F(u, u) = au$

scheme	Order	$\alpha$	$\beta$	CFL condition	DDS bound
Two point upwind	First	$a$	0	$0 < a\lambda \leq 1$	UNO
Three point upwind	Second	$\frac{3a}{2} > 0$	$\frac{a}{2} > 0$	$0 < a\lambda \leq \frac{1}{2}$	$\theta_{i-1}^{n,+} \in \left[-\frac{2-3a\lambda}{a\lambda}, 3\right]$
Beam-Warming	Second	$\frac{a}{2}(3 - \lambda a) > 0$	$\frac{a}{2}(1 - \lambda a) > 0$	$0 < a\lambda < 1$	$\theta_{i-1}^{n,+} \in \left[-\frac{2-a\lambda(3-a\lambda)}{a\lambda(1-a\lambda)}, \frac{3-a\lambda}{1-a\lambda}\right]$
Beam-Warming	Second	$\frac{a}{2}(3 - \lambda a) > 0$	$\frac{a}{2}(1 - \lambda a) < 0$	$1 < a\lambda \leq 2$	$\theta_{i-1}^{n,+} \in \left[-\frac{2-a\lambda(3-a\lambda)}{a\lambda(a\lambda-1)}, \frac{3-a\lambda}{a\lambda-1}\right]$

Table 1: Data dependent stability bounds for three point upwind schemes for  $a > 0$ . Beam-Warming scheme changes its DDS interval with respect to  $CFL$ .

- For  $a > 0$ , the resulting three point conservative difference scheme using flux (7) is

$$u_i^{n+1} = u_i^n - \lambda (\alpha \Delta_- u_i^n - \beta \Delta_- u_{i-1}^n) \quad (8)$$

or

$$u_i^{n+1} = u_i^n - \lambda \left( \alpha - \beta \frac{\Delta_- u_{i-1}^n}{\Delta_- u_i^n} \right) \Delta_- u_i^n \quad (9)$$

By Definition 3.1, approximation (9) is DDS provided

$$0 \leq \lambda(\alpha - \beta \theta_{i-1}^{n,+}) \leq 1.$$

or

$$-\alpha \leq \beta \theta_{i-1}^{n,+} \leq \frac{1 - \alpha\lambda}{\lambda}.$$

which satisfies

$$-\frac{\alpha}{\beta} \leq \theta_{i-1}^{n,+} \leq \frac{1 - \lambda\alpha}{\lambda\beta}, \text{ if } \beta \geq 0 \quad (10a)$$

or

$$\frac{1 - \lambda\alpha}{\lambda\beta} \leq \theta_{i-1}^{n,+} \leq -\frac{\alpha}{\beta}, \text{ if } \beta \leq 0 \quad (10b)$$

- For  $a < 0$ , resulting scheme using flux in (7) can be written as

$$u_j^{n+1} = u_j^n + \lambda \left[ \alpha - \beta \frac{\Delta_+ u_{j+1}^n}{\Delta_+ u_j^n} \right] \Delta_+ u_j^n \quad (11)$$

which by similar calculation is DDS provided

$$-\frac{\alpha}{\beta} \leq \theta_{j+1}^{n,-} \leq \frac{1 - \lambda\alpha}{\lambda\beta}, \text{ if } \beta \geq 0 \quad (12a)$$

$$\frac{1 - \lambda\alpha}{\lambda\beta} \leq \theta_{j+1}^{n,-} \leq -\frac{\alpha}{\beta}, \text{ if } \beta \leq 0 \quad (12b)$$

Note that DDS conditions (10) and (12) gives conditions on the initial data in terms of smoothness parameter  $\theta$  such that local oscillations does not introduced by a three point upwind schemes. In Table 1, Data dependent stability region in terms of  $\theta$  is given for classical upwind schemes.

### 3.2 Three point centred linear schemes

Consider the following generic consistent numerical flux function of three point centred linear scheme for (1)

$$F_{i+\frac{1}{2}} = \alpha u_{i+1}^n + \beta u_i^n, \quad (13)$$

where again for consistency  $\alpha + \beta = a$ .

- $a > 0$  The resulting conservative approximation can be written as

$$u_i^{n+1} = u_i^n - \lambda[\alpha \Delta_+ u_i^n + \beta \Delta_- u_i^n] \quad (14)$$

which using (4) reduces to

$$u_i^{n+1} = u_i^n - \lambda \left( \alpha \theta_i^{n,-} + \beta \right) \Delta_- u_i^n, \quad a > 0 \quad (15)$$

by Definition 3.1, (15) is non-oscillatory stable provided

$$-\beta \leq \alpha \theta_i^{n,-} \leq \frac{1 - \beta \lambda}{\lambda} \quad (16)$$

it reduces to

$$-\frac{\beta}{\alpha} \leq \theta_i^{n,-} \leq \frac{1 - \lambda \beta}{\lambda \alpha}, \quad \alpha > 0 \quad (17a)$$

$$\frac{1 - \lambda \beta}{\lambda \alpha} \leq \theta_i^{n,-} \leq -\frac{\beta}{\alpha}, \quad \alpha < 0 \quad (17b)$$

which on inversion gives condition for (14) to be DDS

$$\theta_i^{n,+} \in \left( -\infty, -\frac{\alpha}{\beta} \right] \cup \left[ \frac{\alpha \lambda}{1 - \beta \lambda}, \infty \right), \quad \alpha > 0 \quad (18a)$$

$$\theta_i^{n,+} \in \left( -\infty, \frac{\lambda \alpha}{1 - \lambda \beta} \right] \cup \left[ -\frac{\alpha}{\beta}, \infty \right), \quad \alpha < 0. \quad (18b)$$

- $a < 0$  scheme (14) can be written as,

$$u_i^{n+1} = u_i^n + \lambda \left( \alpha + \beta \theta_i^{n,+} \right) \Delta_+ u_i^n, \quad a < 0 \quad (19)$$

which using similar calculation is DDS provided

$$\theta_i^{n,-} \in \left( -\infty, -\frac{\beta}{\alpha} \right] \cup \left[ \frac{\lambda \beta}{1 - \lambda \alpha}, \infty \right), \quad \text{if } \beta > 0 \quad (20a)$$

$$\theta_i^{n,-} \in \left( -\infty, \frac{\lambda \beta}{1 - \lambda \alpha} \right] \cup \left[ -\frac{\beta}{\alpha}, \infty \right), \quad \text{if } \beta < 0 \quad (20b)$$

The DDS conditions (18) and (20) classify the data type in terms of the smoothness parameter  $\theta$  such that local oscillations does not get introduced by three point centred schemes. In Table 2, data dependent stability reason is given for classical three point centred schemes which justifies the numerical oscillations by these schemes irrespective of their accuracy. In order to achieve uniformly non-oscillatory (UNO) approximation by an upwind or centred scheme one needs to choose coefficients  $\alpha, \beta$  in numerical flux (7) or (13) such that resulting scheme is DDS  $\forall \theta_{i\pm 1}^{n,\pm} \in \mathbb{R}$ . Bounds in (10) and (12) or (18) and (20) shows that scheme three point upwind or centred are not UNO and will always introduce local numerical oscillation except for the choice  $\alpha = a, \beta = 0$  in (7) or the choice  $\alpha = 0, a > 0$  and  $\beta = 0, a < 0$  in (13). This along with consistency requirement results in to *classical first order upwind scheme*.

scheme	Order	CFL	$\alpha$	$\beta$	DDS bound
Lax-Friedrichs	First	$0 < a\lambda \leq 1$	$\frac{1}{2\lambda}(a\lambda - 1) < 0$	$\frac{1}{2\lambda}(a\lambda + 1) > 0$	$\theta_i^{n,+} \in (\infty, -1] \cup [\frac{1-a\lambda}{1+a\lambda}, \infty)$
FTCS	Second	$0 < a\lambda \leq 1$	$\frac{a}{2} > 0$	$\frac{a}{2} > 0$	$\theta_i^{n,+} \in [-\infty, -1] \cup [\frac{a\lambda}{2-a\lambda}, \infty)$
Lax-Wendroff	Second	$0 < a\lambda \leq 1$	$\frac{a}{2}(1 - a\lambda) > 0$	$\frac{a}{2}(1 + a\lambda) > 0$	$\theta_i^{n,+} \in \left(-\infty, -\frac{1-a\lambda}{1+a\lambda}\right] \cup \left[\frac{a\lambda}{2+a\lambda}, \infty\right)$

Table 2: Data dependent stability bounds for three point centred schemes under CFL  $|a|\lambda \leq 1$

### 3.2.1 Example: A hybrid second order UNO scheme

The second order essentially non-oscillatory ENO scheme [10] for transport problem is UNO under the CFL condition  $0 \leq \lambda a \leq \frac{1}{2}$ .

Proof: Let  $a > 0$ , then the ENO second order reconstruction is

$$\hat{F}_{i+\frac{1}{2}} = \begin{cases} -\frac{a}{2}u_{i-1} + \frac{3a}{2}u_i & \text{if } |u_{i-1} - u_i| < |u_i - u_{i+1}| \\ \frac{a}{2}u_i + \frac{a}{2}u_{i+1} & \text{else} \end{cases}$$

The resulting ENO scheme for the transport problem (1) can be written as

$$u_i^{n+1} = u_i - D\Delta_- u_i \quad (21)$$

where

$$D = \begin{cases} a\lambda \left[ -\frac{1}{2} \frac{\Delta_- u_{i-1}}{\Delta_- u_i} + \frac{3}{2} \right] & \text{if } |\Delta_- u_i| < |\Delta_+ u_i| \text{ and } |\Delta_- u_{i-1}| < |\Delta_- u_i| \\ a\lambda & \text{if } |\Delta_- u_i| < |\Delta_+ u_i| \text{ and } |\Delta_- u_{i-1}| > |\Delta_- u_i| \\ a\lambda \left[ \frac{1}{2} \left( \frac{\Delta_+ u_i}{\Delta_- u_i} - \frac{\Delta_- u_{i-1}}{\Delta_- u_i} \right) + 1 \right] & \text{if } |\Delta_- u_i| > |\Delta_+ u_i| \text{ and } |\Delta_- u_{i-1}| < |\Delta_- u_i| \\ a\lambda \left[ \frac{1}{2} \frac{\Delta_+ u_i}{\Delta_- u_i} + \frac{1}{2} \right] & \text{if } |\Delta_- u_i| > |\Delta_+ u_i| \text{ and } |\Delta_- u_{i-1}| > |\Delta_- u_i| \end{cases}$$

Note under the CFL condition  $0 < \lambda a \leq \frac{1}{2}$ ,  $0 \leq D \leq 1$ , thus scheme is UNO.

## 4 Conclusion

An approach to find non-oscillatory condition on initial data in terms of smoothness parameter is demonstrated for three point schemes. It shows that it is impossible to have a data independent non-oscillatory three point scheme. In future it will be interesting to extend this approach to analyze ENO schemes for their non-oscillatory condition and construction of hybrid high order UNO scheme.

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